

ON EPSTEIN'S ZETA FUNCTION (I)

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1. This paper contains a short account of results whose detailed proofs will be published later.

We define the function $Z(s)$ by

$$Z(s) = \sum' (am^2 + bmn + cn^2)^{-s} \quad (1)$$

where $s = \sigma + it$ (σ and t , real), $\sigma > 1$, and the summation is for all integers m, n (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further a and c are positive numbers while b is real and subject to $4ac - b^2 = \Delta > 0$.

It is well known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole s -plane, and satisfies a functional equation similar to the one satisfied by the Riemann Zeta Function. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

Deuring (*Math. Ztschr.*, **37**, 403–413 (1933)) obtained an important formula for $Z(s)$. Deuring's work led Heilbronn (*Quart. J. Maths., Oxford*, **5**, 150 (1934)) to the proof of the following famous conjecture of Gauss on the class-number of binary quadratic forms with a negative fundamental discriminant: let $h(-\Delta)$ denote the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, then

$$h(-\Delta) \rightarrow \infty \quad \text{as} \quad \Delta \rightarrow \infty \quad (2)$$

Again using the ideas of Heilbronn and Deuring, Siegel proved that

$$h(-\Delta) > \Delta^{1/2 - \epsilon} \quad [\Delta > \Delta_0(\epsilon)] \quad (3)$$

which is a great advance on (2).

Our starting point is the formula:

$$Z(s) = 2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\Delta^{s-1/2}}\zeta(2s-1)\Gamma(s-1/2) + Q(s) \quad (4)$$

where

$$Q(s) = \frac{\pi^s \cdot 2^{s+1/2}}{a^{1/2}\Gamma(s)\Delta^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2} \sigma_{1-2s}(n) \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \phi^s \phi^{-1/2} \exp\left\{-\frac{\pi n \Delta^{1/2}}{2a}(\phi + \phi^{-1})\right\} d\phi \quad (4)$$

Here $\sigma_k(n)$ denotes the sum of the k th powers of the divisors of n , and $\zeta(s)$ is Riemann's Zeta Function. The series for $Q(s)$ is highly convergent. Taking a crude estimate of the series for $Q(s)$ we obtain the formula of Deuring referred to above.

2. The formula (4) can be applied to the proof of the positiveness of certain Dirichlet L -functions at $s = 1/2$. In fact we define for $s > 0$,

$$L_p(s) = \sum_1^{\infty} \left(\frac{n}{p} \right) n^{-s}$$

where (n/p) is Legendre's symbol defined as follows:

If $n \not\equiv 0 \pmod{p}$, then $(n/p) = +1$ if the congruence $x^2 \equiv n \pmod{p}$ is soluble; $(n/p) = -1$ if the congruence $x^2 \equiv n \pmod{p}$ is insoluble

If $n \equiv 0 \pmod{p}$, then

$$(n/p) = 0$$

The positiveness of $L_p(s)$ for $0 < s \leq 1$ was proved by S. Chowla (*Acta Arithmetica*, Band 1, 114 (1935)) in a large number of special cases, e.g., for $p = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 47, 53, 59, 61, 71, 73, 79, 83, 89, 97$.

But no information was obtained in the cases $p = 43, 67, 163$ (here the class number $h(-p)$ is small). Heilbronn (*Acta Arithmetica*, Band 2, 212 (1937)) proved that there are infinitely many primes p for which the method of Chowla gives no information. Curiously enough, the present method is more successful with precisely those cases like $p = 43, 67, 163$ (class number $h(-p) = 1$) where the previous method failed. In these three cases we obtain $L_p(1/2) > 0$ (Rosser has recently, in an unpublished paper, settled the cases $p = 43$ and $p = 67$ by an entirely different method). That $L_p(1/2) > 0$ in these cases, is not surprising, for if there is a prime p such that $L_p(1/2) < 0$ then the extended Riemann hypothesis is false! These results are deduced from the following

THEOREM: If p is an odd prime > 7 and if $h(-p) = 1$, then ($c = \pi/2$)

$$\zeta(1/2)L_p(1/2) = \gamma + \log \left(\frac{\sqrt{p}}{8\pi} \right) + \frac{8\theta \cdot e^{-c\sqrt{p}}}{\pi\sqrt{p}(1 - e^{-c\sqrt{p}})} \quad (5)$$

where γ is Euler's constant and θ is a real number such that $|\theta| < 1$. Remark that we can also show the positivity of $L_p(\sigma)$ on the whole stretch $1/2 \leq \sigma \leq 1$ by the same method, in the three cases $p = 43, 67, 163$. This can be done with a little more computation.

3. It is well known that we have

$$h(-d) = 1 \quad (6)$$

in the nine cases $d = 3, 4, 7, 8, 11, 19, 43, 67, 163$. Heilbronn and Linfoot

have proved that (6) has at most 10 solutions; further, that if (6) has a tenth solution then d must be very large indeed; in fact $d > 5.10^9$ (Lehmer).

It follows from (5) that if (6) has a tenth solution $d = d_0$, then

$$L_d(1/2) < 0 \quad [d = d_0]$$

It is known that d_0 is necessarily a prime, and is in fact, $\equiv 3 \pmod{8}$.

4. We apply (4) to a classical problem of the theory of elliptic functions. Write, as usual,

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad (0 < k < 1)$$

$$K' = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}} \quad (k^2 + k'^2 = 1)$$

It has long been known that K can be calculated in finite terms whenever iK'/K is a number belonging to any of the imaginary quadratic fields $k(\sqrt{-1})$, $k(\sqrt{-2})$, $k(\sqrt{-3})$. This is deduced from the fact that K can be calculated in finite terms when

$$K'/K = \sqrt{n} \quad (n = 1, 2, 3)$$

Thus when $n = 1$ i.e., $k = \frac{1}{\sqrt{2}}$, we have

$$K = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}$$

and there are similar results obtained (each by a different method!) for the cases $n = 2, 3$. We prove that K can be calculated in finite terms whenever iK'/K is a number in an imaginary quadratic field. More precisely, our result is as follows: let d be a negative integer $\equiv 0$ or $1 \pmod{4}$ and so that d or $d/4$ is a square-free integer. Further let h denote the class-number $h(d)$ and

$$w = 6, 4, 2 \quad \text{according as } d = -3, d = -4, d < -4.$$

Finally (d/m) denotes the Kronecker symbol. Then,

THEOREM: Let iK'/K be a number from the field $k(\sqrt{d})$. Then we have

$$K = \lambda \sqrt{\pi} \left\{ \prod_{m=1}^{|d|} \pi \Gamma \left(\frac{m}{|d|} \right)^{(d/m)} \right\}^{w/4h} \quad (7)$$

where λ is an algebraic number.

A special case of (7) is the following:

THEOREM: If $K'/K = \sqrt{p}$ and if $h(-p) = 1$, then

$$\frac{2K}{\pi} = \frac{2^{1/4}(kk')^{-1/4}}{\sqrt{2\pi p}} \left\{ \frac{\pi \Gamma\left(\frac{\alpha}{p}\right)}{\pi \Gamma\left(\frac{\beta}{p}\right)} \right\}^{w/4} \quad (8)$$

where (p is a prime) and $w = 6$ if $p = 3$, $w = 2$ if $p > 3$; α runs through the $\frac{p-1}{2}$ quadratic residues of p that lie between 0 and p , while β runs through the remaining $\frac{p-1}{2}$ numbers between 0 and p .

Specializing again to the case $p = 7$ we obtain in the usual notation for hypergeometric series:

$$F(1/4, 1/4, 1; 1/64) = \sqrt{\frac{2}{7\pi}} \left\{ \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)} \right\}^{1/2}$$

5. Let $G_d(s)$ denote the analytical continuation of the function defined for $\sigma > 3/2$ by the series

$$\sum' (x^2 + y^2 + dz^2)^{-s}$$

From a formula similar to (4) it is deduced that

THEOREM: *There exists a real number θ_d such that*

$$G_d(\theta_d) = 0 \quad [d > d_0]$$

where $\theta_d \rightarrow 0$ as $d \rightarrow \infty$, but $\theta_d \neq 0$.

ON A NEW METHOD IN ELEMENTARY NUMBER THEORY WHICH LEADS TO AN ELEMENTARY PROOF OF THE PRIME NUMBER THEOREM

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1. *Introduction.*—In the course of several important researches in elementary number theory A. Selberg¹ proved some months ago the following asymptotic formula:

$$\sum_{p \leq x} (\log p)^2 + \sum_{pq \leq x} \log p \log q = 2x \log x + O(x), \quad (1)$$

where p and q run over the primes. This is of course an immediate consequence of the prime number theorem. The point is that Selberg's in-